Nonlinear transient dynamics of the driven Klein-Gordon solitons

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The transient nonlinear dynamics of single sine-Gordon and ϕ^4 kinks in a small constant external field is investigated analytically. Besides the acceleration of the mass center, the energy supplied by the field is shown to be accumulated by transient oscillations of the kink width: the frequency of oscillation of the kink width grows with the field and time while its amplitude decreases. The dynamics in the asymptotic region illustrates the unstable nature of the kinks. The ϕ^4 kink is shown to be much more sensitive to the external field than the sine-Gordon one.

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The dynamics of the driven Klein-Gordon solitons in a constant external field shows a highly transient nonlinear behavior [1,2]. It has been shown that the linear (WKB) perturbation scheme for the solution of the respective nonlinear equations is insufficient for the driven solitons [1]: a one-dimensional nucleation process due to the cooperation of the nonlinearity and of a small constant external driving field generates new nonlinear traveling oscillations with the amplitude proportional to the external field. The scenario of the nonlinear transient dynamics yields the following time development: the nonlinear fluctuations of the (shifted) ground state accumulate the energy supplied by the external field until there is enough energy for the creation of a kink. This process of the accumulation of energy continues by the acceleration of the kink up to a critical velocity accompanied by the decrease of the soliton width and simultaneously by the generation of the nonlinear oscillations of the kink moving in the opposite direction until a new kink is created, etc.

The transient dynamics of a single kink becomes more transparent when described by the collective coordinates: the coordinate of the mass center $X(t)$ and the width of the kink $L(t)$ are coupled yielding a highly nonlinear differential equation for $L(t)$. In the paper [1] we solved the dynamic equation for $L(t)$ by separation of the slowly and rapidly time dependent terms with the restriction to the region of small times. However, in contrast to the case with the damping where the supply of energy by the external field is compensated by the transfer of the energy to a reservoir [3] leaving the total energy constant so that the transient regime is determined by the interplay of the damping and of the external field yielding nonlinear transient relaxation, in the present case the transient nature of the dynamics is not limited by time. That means that the time development of the kink keeps its transient nature during its whole life.

The aim of this paper is to find characteristics of the nonlinear dynamics of the collective coordinate of kinks. We shall solve the dynamic equations exactly and discuss their physical consequences.

It was shown by Rice [4] that both sine-Gordon and ϕ^4 models can be described by the same set of equations for the collective coordinates $X(t)$ and $L(t)$ with different constant parameters for each of the models. The resulting oscillations of the width of the sine-Gordon soliton exhibit a mode which was not found by the perturbation approach to the sine-Gordon equation. This is a consequence of the fact that Rice's equations do not take into account the phonon excitations in analytical calculations. An exact collective coordinate formalism including phonon modes has been elaborated by Boesch and Willis [5]. They have found the above mentioned mode unstable as it entered the phonon band.

In what follows we shall confine ourselves to Rice's formalism. We shall solve respective dynamic equations for the collective coordinates exactly and for any time. They read as follows [1]:

$$\frac{dp_X}{dt} = 2\pi f \ ,$$

$$\frac{dp_L}{dt} = -\frac{1}{2m_s L_0}(p_L^2/\alpha + p_X^2) - \frac{dU(L)}{dL}, \qquad (1)$$

where

$$p_X(t) = \frac{m_s L_0}{L(t)}\dot{X}(t), \ , p_L(t) = \alpha\frac{m_s L_0}{L(t)}\dot{L}(t) \ ,$$

$$U(L) = \frac{E_s}{2}[L_0/L(t) + \cos\phi_s L(t)/L_0] \ , \qquad (2)$$

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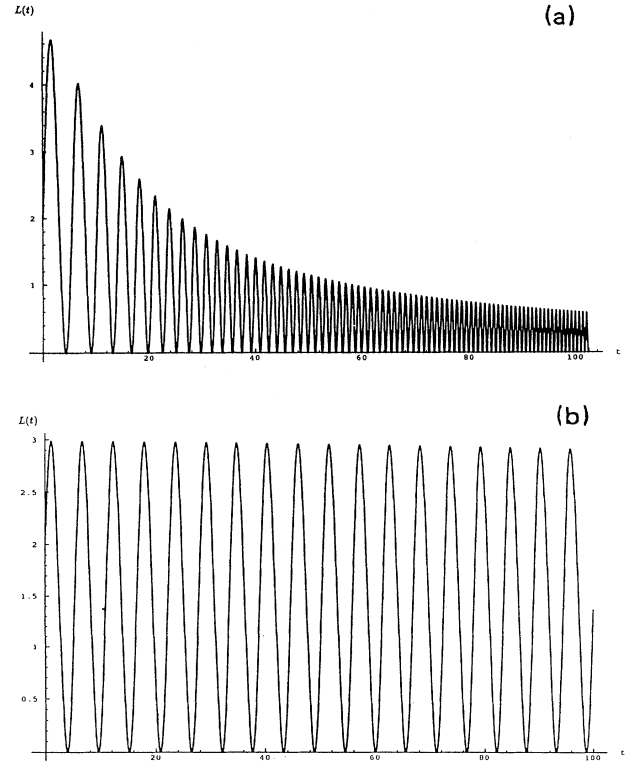

FIG. 1. Exact time dependence of $L(t)$ given by (14) for the sine-Gordon soliton with (a) $f = 0.1$, (b) $f = 0.002$, $c_0 = 0.1$.

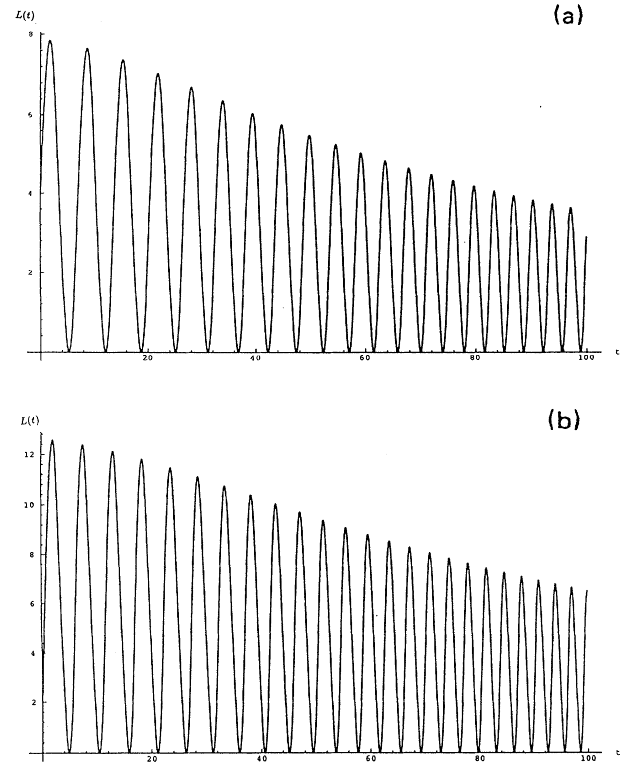

FIG. 2. The same for the ϕ^4 case with (a) $f = 0.02$, (b) $f = 0.002$.

and

$$m_s = c_0^{-2} E_s, \quad L_0 = L(0) \ ,$$

with $\phi_s = -\arcsin f$, $E_s = 8c_0$, $L_0 = 2(1 - \frac{v_0^2}{c_0^2})^{1/2}$, $\alpha = \pi^2/48$ for the sine-Gordon case, $V(\phi) = 1 - \cos\phi$; and $E_s = (\frac{2}{3})c_0$, $L_0 = 4(1 - \frac{v_0^2}{c_0^2})^{1/2}$, $\alpha = (\pi^2 - 6)/48$ for the ϕ^4 case, $V(\phi) = 1/8(1 - \phi^2)^2$ [3].

Using Eqs. (1) and (2) we get [1]

$$\frac{d}{dt}\left(\frac{\dot{L}}{L}\right) + \frac{1}{2}\left(\frac{\dot{L}}{L}\right)^2 + \frac{c_0^2}{2\alpha}\left(\frac{\cos\phi_s}{L_0^2} - \frac{1}{L^2}\right) + 2\beta^2\left(t + \frac{p_0}{2\pi f}\right)^2 = 0, \qquad (3)$$

where $p_0 = p_X(0)$ is an initial momentum and $\beta = \pi f/\alpha^{1/2} m_s L_0$.

Further, using the ansatz $L(t) = g^2(t)$ Eq. (3) becomes

$$\ddot{g} + \{\Omega^2 + \beta^2[t^2 + tp_0/(\pi f)]\}g - \frac{c_0^2}{4\alpha g^3} = 0, \qquad (4)$$

where $\Omega^2 = c_0^2/4(\alpha L_0^2)\left[\cos\phi_s + p_0^2/(m_s^2 c_0^2)\right]$. Equation (4) can be solved exactly: according to Pinney [6] the solution to Eq. (4) can be written in the form

$$g(t) = \left[u^2(t) + \frac{c_0^2}{4\alpha} W^{-2} v^2(t)\right]^{1/2}, \qquad (5)$$

where W is Wronskian $W = u'v - uv'$, u, v is the fundamental set of solutions of the respective linear equation

$$\ddot{y}(t) + [\bar{\Omega}^2 + \beta^2 \bar{t}^2]y = 0, \qquad (6)$$

where we introduced

$$\bar{\Omega}^2 = \Omega^2 - \frac{\beta^2 p_0^2}{4\pi^2 f^2}, \quad \bar{t} = t + \frac{p_0}{2\pi f}.$$

Equation (6) can be rewritten when using an ansatz

$$y(x) = x^{-1/4} Y(x), \quad x = \frac{1}{2}\bar{t}^2, \qquad (7)$$

where $Y(x)$ obeys the equation

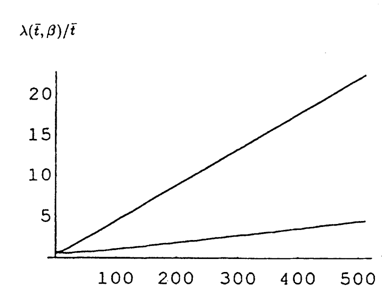

FIG. 3. Time dependence of the frequency of oscillations $\lambda(\bar{t}, \beta)/\bar{t}$, for $f = 0.02$ and $f = 0.1$. [See Eq. (17).]

$$Y'' + \left(\frac{3}{16}\frac{1}{x^2} + \frac{\bar{\Omega}^2}{2}\frac{1}{x} + \beta^2\right) Y = 0. \tag{8}$$

Solutions to this equation are Whittaker's functions [7]

$$Y(x) = Y\left(\frac{-i\bar{\Omega}^2}{4\beta}, \pm\frac{1}{4}; \rho x\right) \equiv M_{k,\pm m}(i\beta\bar{t}^2). \tag{9}$$

Here, $k = \frac{-i\bar{\Omega}^2}{4\beta}, m = \frac{1}{4}, \rho = i2\beta$, and $M_{k,\pm m}(i\beta t^2)$ are defined as

$$M_{k,\frac{1}{4}}(z) = z^{3/4}\exp(-\tfrac{1}{2}z)\phi(\tfrac{3}{4} - k, \tfrac{3}{2}; z) ,$$

$$M_{k,\frac{-1}{4}}(z) = z^{1/4}\exp(-\tfrac{1}{2}z)\phi(\tfrac{1}{4} - k, \tfrac{1}{2}; z) , \tag{10}$$

where $\phi(a, b; z)$ is a degenerated hypergeometric function and $z = i\beta\bar{t}^2$.

Then, the general solution to the linear equation (6) is given as a linear combination of two independent solutions given by (7), (9), and (10),

$$y_i = \left(\frac{\sqrt{2}}{\bar{t}}\right)^{1/2} [A_i M_{k,\frac{1}{4}}(i\beta\bar{t}^2) + B_i M_{k,\frac{-1}{4}}(i\beta\bar{t}^2)], \quad i = 1, 2. \tag{11}$$

The respective Wronskian W is constant, $W = \frac{1}{2}(A_2B_1 - A_1B_2)$. As $M_{k,\pm m}(i\beta\bar{t}^2)$ determine linearly independent solutions of Eq. (8) we choose $A_2 = B_1 = 0$. For $L(t) = g^2(t) = \sqrt{2}/\bar{t}Y^2(t)$ we get finally

$$L(\bar{t}, \beta) = \frac{\sqrt{2}}{\bar{t}}\left(A_1^2 M^2_{k,\frac{1}{4}}(i\beta\bar{t}^2) + \frac{c_0^2}{\alpha}\frac{1}{A_1^2} M^2_{k,-\frac{1}{4}}(i\beta\bar{t}^2)\right). \tag{12}$$

A_1 can be determined from the initial condition $L(t = 0) = L_0$ for $\bar{t} = t_0 = p_0/2\pi f \neq 0$, as

$$A_1^2(t_0) = \frac{L_0}{(2i\beta)^{3/2}t_0^2}\frac{1}{\phi_1^2(t_0)}\left(1 + \left[1 + \frac{8c_0^2\beta^2t_0^2}{L_0^2} \times\exp(-2i\beta t_0^2)\Phi_1^2(t_0)\phi_2^2(t_0)\right]^{1/2}\right). \tag{13}$$

With the use of (13) we get finally an exact solution for $L(\bar{t}, \beta)$,

$$L(\bar{t}, \beta) = \frac{L_0}{2}\exp[-i\beta(\bar{t}^2 - t_0^2)]\left(\frac{\phi_2^2(\bar{t})}{\phi_2^2(t_0)} + \frac{\bar{t}^2}{t_0^2}\frac{\phi_1^2(\bar{t})}{\phi_1^2(t_0)} + \left\{1 + 2\left[\frac{2c_0\beta t_0}{L_0}\exp(-i\beta t_0^2)\phi_1(t_0)\phi_2(t_0)\right]^2\right\}^{1/2} \times\left[-\phi_2^2(\bar{t})/\phi_2^2(t_0) + \frac{\bar{t}^2}{t_0^2}\phi_1^2(\bar{t})/\phi_1^2(t_0)\right]\right). \tag{14}$$

Here, we use notations

$$\phi_1(t) \equiv \phi(\tfrac{3}{4} - k, \tfrac{3}{2}; i\beta t^2) ,$$

$$\phi_2(t) \equiv \phi(\tfrac{1}{4} - k, \tfrac{1}{2}; i\beta t^2). \tag{15}$$

In Figs. 1 and 2 there are plotted the functions $L(\bar{t}, \beta)$ given by (14) for both the sine-Gordon and ϕ^4 case for different values of f. (The numerical results were obtained using the program MATHEMATICA [8].) From comparison of the dynamics of both cases it results that the ϕ^4 kink is much more sensitive to the external field than the sine-Gordon one. The amplitudes of oscillations and their frequencies are both field and time dependent.

Further, we shall find analytical expressions for asymptotic behavior of the kinks. As our considerations of the soliton dynamics are restricted for small f, $f \ll 1$, (i.e. for small β), we shall use the asymptotic expansion of the Whittaker's functions for large $| k |= \frac{\bar{\Omega}^2}{4\beta} \gg 1$, and $\beta\bar{t}^2 \gg 1$ while $| k | /\beta\bar{t}^2 = \left(\bar{\Omega}/2\beta\bar{t}\right)^2$ is finite [7]. This yields

$$M_{-i\frac{\bar{\Omega}^2}{4\beta},\frac{1}{4}}(i\beta\bar{t}^2) \to \exp\left(i\pi\frac{3}{8}\right)\left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^{1/2}\left(\frac{\beta}{\bar{\Omega}^2 + (\beta\bar{t})^2}\right)^{\frac{1}{4}}\sin[\lambda(\bar{t}, \beta)]\left[1 + O\left(\frac{2}{\bar{\Omega}}\sqrt{\beta}\right)\right], \tag{16a}$$

$$M_{-i\frac{\bar{\Omega}^2}{4\beta},\frac{-1}{4}}(i\beta\bar{t}^2) \to \exp\left(i\pi\frac{1}{8}\right)(\beta\bar{t})^{1/2}\left\{\beta\left[1 + \left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]\right\}^{-1/4}\sin[\lambda(\bar{t}, \beta)]\left[1 + O\left(\frac{2}{\bar{\Omega}}\sqrt{\beta}\right)\right], \tag{16b}$$

where

$$\lambda(\bar{t}, \beta) = \frac{\bar{\Omega}\bar{t}}{2}\left[1 + \left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]^{1/2} + \frac{\bar{\Omega}^2}{2\beta}\text{arcsinh}\left\{\frac{2\beta\bar{t}}{\bar{\Omega}}\left[1 + \left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]^{1/2}\right\}. \tag{17}$$

Finally, the asymptotic behavior of Whittaker's functions (16a) and (16b) yields for the asymptotics of the hypergeometric functions $\phi_1(\bar{t})$ and $\phi_2(\bar{t})$ the following:

$$\phi_1(\bar{t}) \to \phi_{1as}(\bar{t}) = (\bar{t}\bar{\Omega})^{-1}\exp\left(\frac{i\beta\bar{t}^2}{2}\right)\left[1+\left(\frac{\beta\bar{t}}{2\bar{\Omega}}\right)^2\right]^{\frac{-1}{4}}$$
$$\times\sin\left(\frac{\bar{\Omega}\bar{t}}{2}\left[1+\left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]^{1/2}+\frac{\bar{\Omega}^2}{2\beta}\text{arcsinh}\left\{\frac{2\beta\bar{t}}{\bar{\Omega}}\left[1+\left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]^{1/2}\right\}\right) \tag{18a}$$

and

$$\phi_2(\bar{t}) \to \phi_{2as}(\bar{t}) = \bar{t}\bar{\Omega}\phi_{1as}(\bar{t}), \tag{18b}$$

where $\phi_{1as}(\bar{t})$ is given by (18a).

Finally, with the use of expressions (14) and (18) we get for the asymptotic behavior of the width of the soliton $L_{as}(\bar{t},\beta)$ the result

$$L_{as}(\bar{t},\beta) = C(t_0,\beta)\left[1+\left(\frac{\beta\bar{t}}{\bar{\Omega}}\right)^2\right]^{-1/2}\left[1-\cos 2\lambda(\bar{t},\beta)\right]. \tag{19}$$

Here, C is a constant, given by the initial condition

$$C(t_0,\beta) = \frac{L_0}{4}\exp(i\beta t_0^2)\left\{\phi_2^{-2}(t_0)+(\Omega t_0)^{-2}\phi_1^{-2}(t_0)\right.$$
$$\left.+[(\Omega t_0)^{-2}\phi_1^{-2}(t_0)-\phi_2^{-2}(t_0)]\left[1+2\left(\frac{2c_0\beta t_0}{L_0}\exp(-i\beta t_0^2)\phi_1(t_0)\phi_2(t_0)\right)^2\right]^{1/2}\right\}.$$

$L_0 = L(0)$ was defined below Eq. (2). The asymptotic dependence $L_{as}(\bar{t},\beta)$ (19) was derived for $|k| = \frac{\Omega^2}{4\beta}$ large [f is small, $\beta \sim f$, see Eq. (3)] and for $\beta\bar{t}$ finite, i.e., for times $\bar{t} \sim f^{-1}$. Then, the dominating term in $\lambda(\bar{t},\beta)$, (17), is the first one, $\lambda(\bar{t},\beta) \sim \frac{\bar{\Omega}\bar{t}}{2}[1+(\frac{\beta\bar{t}}{\bar{\Omega}})^2]^{1/2}$. The frequency of oscillations of $L_{as}(\bar{t},\beta)$ (19) is then

$$\omega_{as}(\bar{t},\beta) = 2\lambda(\bar{t},\beta)/\bar{t} = \left[\bar{\Omega}^2+(\beta\bar{t})^2\right]^{1/2}. \tag{20}$$

According to Eq. (5) the effect of nonlinearity for the kink width is manifested by a shift of the width (nonoscillating) and by a doubling of the frequency of linear oscillators (6). The asymptotic frequency (20) is identical with the frequency of the linear oscillations in Eq. (6). Hence, the effects of nonlinearity on the frequency of oscillations decrease with increasing time. However, the range of stability of these oscillations is limited in time as at certain times the respective bound energy level (ϕ^4 case) enters the phonon band.

The frequency $\lambda(\bar{t},\beta)/\bar{t}$ given by Eq. (17) is plotted in Fig. 3. The effect of nonlinearity is evident for small times again; the slope of the linear behavior in the asymptotic region is strongly influenced by the external field f as expected.

In the future we intend to compare our results with the numerical simulations of the solution of the sine-Gordon equation with a constant external field.

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