Supplementary Information

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1 Analytical derivation of Eq. in the main text

Let us introduce two dimensionless parameters

$$h \equiv \lambda_c/H \quad \text{and} \quad q \equiv k/K\,, \tag{1}$$

so that $K_L = K/h$, $K_T = Kh/2(1+\sigma)$, $\beta = 1/(1+b)$, $1-\beta = b/(1+b)$, $(\lambda_c\kappa_T)^2 = hq/b$, $(\lambda_c\kappa)^2 = q\,(1+b)/b$,

$$\varepsilon \equiv \frac{\kappa_T^2}{\kappa^2} = \frac{h}{1+b}\,, \tag{2}$$

where $b = 2(1+\sigma)q/h$, and consider the typical system with $h, q \ll 1$. In this case $\varepsilon \ll 1$, so that $\kappa_1^2 \approx \kappa^2(1+\beta\varepsilon)$ and $\kappa_2^2 \approx \kappa^2\varepsilon\,(1-\beta) = q\varepsilon/\lambda_c^2$, or

$$(\lambda_c\kappa_2)^{-1} \approx [2(1+\sigma) + h/q]^{1/2}/h\,. \tag{3}$$

In accordance with the numerics (see Fig. 2a), let us assume that in the case of $h, q \ll 1$ the displacement field in the TB is given by

$$u_t(x) \approx U_{t0}e^{-\kappa_2 x} \tag{4}$$

and does not change during front propagation. Before nucleation of the first precursor, the solution of Eq. 7 in the main text is

$$\begin{aligned} u(x) &= A_{30}\sinh(\kappa x) + A_{40}\cosh(\kappa x) \\ &- \kappa\beta U \int_0^x d\xi\, e^{-\kappa_2\xi}\sinh[\kappa(x-\xi)] \\ &= \frac{1}{2}\left(A_{40} + A_{30} - \frac{1}{2}\beta U\frac{\kappa}{\kappa+\kappa_2}\right)e^{\kappa x} \\ &+ \frac{1}{2}\left(A_{40} - A_{30} - \beta U\frac{\kappa}{\kappa-\kappa_2}\right)e^{-\kappa x} \\ &+ \beta U\frac{\kappa^2}{\kappa^2-\kappa_2^2}e^{-\kappa_2 x}. \end{aligned} \tag{5}$$

The right-hand-side boundary condition, $u(x) \to 0$ at $x \to \infty$, gives us

$$A_{40} + A_{30} = \frac{1}{2}\,\beta U \frac{\kappa}{\kappa + \kappa_2}\,, \tag{6}$$

while the left-hand-side boundary condition (Eq. 10 in the main text) leads to the equation

$$(A_{40} - A_{30})(1 + \lambda_c \kappa)(\kappa + \kappa_2) = \beta U \kappa (1 + a\kappa + 2\lambda_c \kappa_2)\,. \tag{7}$$

Thus, before nucleation of the first precursor, the IL displacement field is

$$u(x) = \frac{\beta U \kappa^2}{(\kappa^2 - \kappa_2^2)} \left(e^{-\kappa_2 x} - \frac{\kappa_2}{\kappa} \frac{(1 + \lambda_c \kappa_2)}{(1 + \lambda_c \kappa)}\, e^{-\kappa x} \right). \tag{8}$$

Equation (8) allows us to couple the parameters $U \equiv u_t(0)$ and $u_c \equiv u(0)$:

$$U = u_c \left(1 + \frac{\kappa_2}{\kappa}\right) / (\beta \Psi_1)\,, \tag{9}$$

where

$$\Psi_1 = 1 + \frac{\kappa_2}{\kappa} \frac{\lambda_c \kappa}{(1 + \lambda_c \kappa)}\,. \tag{10}$$

When the displacement of the IL trailing edge reaches the threshold value u_s at some $U = U_0 = u_s(1 + \kappa_2/\kappa)/(\beta\Psi_1)$, the front starts to propagate. In this case the solution of Eq. 7 in the main text, ahead of the propagating front, $x > s$, where $u_b(x) = 0$ so that $w(x) = \beta u_t(x) = \beta U_0 e^{-\kappa_2 x}$, is given by

$$\begin{aligned} \widetilde{u}(x; s) &= A_3(s)\, e^{-\kappa(x-s)} + A_4(s)\, e^{\kappa(x-s)} \\ &- \kappa \beta U_0 \int_s^x d\xi\, e^{-\kappa_2 \xi} \sinh[\kappa(x - \xi)] \\ &= \frac{\beta U_0 \kappa^2 e^{-\kappa_2 x}}{(\kappa^2 - \kappa_2^2)} + A_3(s)\, e^{-\kappa(x-s)} + A_4(s)\, e^{\kappa(x-s)} \\ &- \frac{1}{2}\,\beta U_0 \kappa\, e^{-\kappa_2 s} \left[\frac{e^{\kappa(x-s)}}{(\kappa + \kappa_2)} + \frac{e^{-\kappa(x-s)}}{(\kappa - \kappa_2)} \right]. \end{aligned} \tag{11}$$

The right-hand-side boundary condition gives us the coefficient $A_4(s)$,

$$A_4(s) = \frac{1}{2}\,\beta U_0 \frac{\kappa\, e^{-\kappa_2 s}}{(\kappa + \kappa_2)}\,, \tag{12}$$

so that Eq. (11) takes the form

$$\begin{aligned} \widetilde{u}(x; s) &= \frac{\beta U_0 \kappa^2}{(\kappa^2 - \kappa_2^2)}\, e^{-\kappa_2 x} \\ &+ \left[A_3(s) - \frac{1}{2}\,\beta U_0 \frac{\kappa\, e^{-\kappa_2 s}}{(\kappa - \kappa_2)} \right] e^{-\kappa(x-s)}. \end{aligned} \tag{13}$$

Behind the propagating front, $x < s$, where $w(x) = \beta\, u_t(x) + (1-\beta)\, u_b(x)$ and $u_b(x) = \widetilde{u}(x+0;x) = A_3(x) + A_4(x)$, the solution of Eq. 7 in the main text is given by

$$
\begin{aligned}
&\widetilde{u}(x;s) = A_1(s)\sinh(\kappa x) + A_2(s)\cosh(\kappa x) \\
&- \kappa\,(1-\beta)\int_0^x d\xi\,[A_3(\xi)+A_4(\xi)]\,\sinh[\kappa(x-\xi)] \\
&- \kappa\beta U_0 \int_0^x d\xi\, e^{-\kappa_2\xi}\sinh[\kappa(x-\xi)] \\
&= \beta U_0\mathcal{F}(x) + A_1(s)\sinh(\kappa x) + A_2(s)\cosh(\kappa x) \\
&- \kappa\,(1-\beta)\int_0^x d\xi\, A_3(\xi)\,\sinh[\kappa(x-\xi)]\,,
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
&\mathcal{F}(x) = \frac{\Psi_2\kappa^2}{(\kappa^2-\kappa_2^2)} \\
&\times\left(\frac{\kappa_2}{\kappa}\sinh(\kappa x) - \cosh(\kappa x) + e^{-\kappa_2 x}\right), && (15)\\
&\frac{\mathcal{F}'(x)}{\kappa} = \frac{\Psi_2\kappa^2}{(\kappa^2-\kappa_2^2)} \\
&\times\left(\frac{\kappa_2}{\kappa}\cosh(\kappa x) - \sinh(\kappa x) - \frac{\kappa_2}{\kappa}\,e^{-\kappa_2 x}\right), && (16)\\
&\Psi_2 = \frac{(3-\beta)\kappa + 2\kappa_2}{2(\kappa+\kappa_2)}\,, && (17)
\end{aligned}
$$

so that $\mathcal{F}(0) = 0$ and $\mathcal{F}'(0) = 0$.

The coefficients $A_{\ldots}(s)$ in these equations are determined by the boundary and continuity conditions. The left-hand-side boundary condition (Eq. 10 in the main text) couples the coefficients $A_1(s)$ and $A_2(s)$. Using $u_b(0) = u_s$, $\widetilde{u}(0;s) = A_2(s)$ and $\widetilde{u}'(0;s) = \kappa A_1(s)$, we obtain

$$
\begin{aligned}
&A_2(s) - (\lambda_c\kappa)^{-1}A_1(s) = \Psi_3\,,\\
&\Psi_3 = \beta U_0 + (1-\beta)\,u_s\,.
\end{aligned}
\tag{18}
$$

The continuity conditions (Eqs. 23 and 24 in the main text) lead to two equations

$$
\begin{aligned}
&\kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\,\sinh[\kappa(s-\xi)] + A_3(s) \\
&\quad = A_1(s)\sinh(\kappa s) + A_2(s)\cosh(\kappa s) + \beta U_0\Psi_4(s)
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&\kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\,\cosh[\kappa(s-\xi)] - A_3(s) \\
&\quad = A_1(s)\cosh(\kappa s) + A_2(s)\sinh(\kappa s) + \beta U_0\Psi_5(s)\,,
\end{aligned}
\tag{20}
$$

where

$$\Psi_4(s) = \mathcal{F}(s) - \frac{\kappa\, e^{-\kappa_2 s}}{2(\kappa+\kappa_2)}\ , \tag{21}$$

$$\Psi_5(s) = \frac{\mathcal{F}'(s)}{\kappa} - \frac{\kappa\, e^{-\kappa_2 s}}{2(\kappa+\kappa_2)}\ . \tag{22}$$

Taking the difference and sum of Eqs. (19) and (20), we obtain two new equations:

$$\begin{aligned} &2A_3(s)\, e^{\kappa s} - \kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\, e^{\kappa\xi} \\ &= A_2(s) - A_1(s) + \beta U_0[\Psi_4(s) - \Psi_5(s)]\, e^{\kappa s}, \end{aligned} \tag{23}$$

$$\begin{aligned} &\kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\, e^{-\kappa\xi} \\ &= A_2(s) + A_1(s) + \beta U_0\,[\Psi_4(s) + \Psi_5(s)]\ e^{-\kappa s}. \end{aligned} \tag{24}$$

Using Eq. (18), Eqs. (21) and (22) may be rewritten as

$$\begin{aligned} 2A_3(s)\, e^{\kappa s} - \kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\, e^{\kappa\xi} &= A_1(s)\frac{(1-\lambda_c\kappa)}{\lambda_c\kappa} \\ &+ \Psi_3 + \beta U_0[\Psi_4(s) - \Psi_5(s)]\, e^{\kappa s}, \end{aligned} \tag{25}$$

$$\begin{aligned} \kappa\,(1-\beta)\int_0^s d\xi\, A_3(\xi)\, e^{-\kappa\xi} &= A_1(s)\frac{(1+\lambda_c\kappa)}{\lambda_c\kappa} \\ &+ \Psi_3 + \beta U_0\,[\Psi_4(s) + \Psi_5(s)]\ e^{-\kappa s}. \end{aligned} \tag{26}$$

Combining these equations, we finally come to the integral equation for the coefficient $A_3(s)$:

$$\begin{aligned} &A_3(s)(1+\lambda_c\kappa)\, e^{\kappa s} \\ &- \kappa(1-\beta)\int_0^s d\xi\, A_3(\xi)\,[\cosh(\kappa\xi) + (\lambda_c\kappa)\sinh(\kappa\xi)] \\ &= \lambda_c\kappa\Psi_3 + \beta U_0\Psi_2\Psi_6(s)\,, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Psi_6(s) = &\frac{\kappa\,(1+\lambda_c\kappa)}{2(\kappa-\kappa_2)}\, e^{(\kappa-\kappa_2)s} \\ &- \frac{\kappa\,(\lambda_c\kappa^2+\kappa_2)}{(\kappa^2-\kappa_2^2)}\left[1 + e^{-(\kappa+\kappa_2)s}\right] \end{aligned} \tag{28}$$

so that

$$\Psi_6'(s) = \left[\frac{(1+\lambda_c\kappa)}{2}\, e^{\kappa s} + \frac{(\lambda_c\kappa^2+\kappa_2)}{(\kappa-\kappa_2)}\, e^{-\kappa s} \right] \kappa\, e^{-\kappa_2 s}. \qquad (29)$$

From Eq. (27) we find that

$$A_3(0) = [\lambda_c\kappa\Psi_3 + \beta U_0 \Psi_2 \Psi_6(0)]/(1+\lambda_c\kappa)\,. \qquad (30)$$

Differentiating Eq. (27), we obtain a differential equation for $A_3(s)$:

$$\begin{aligned} \frac{1}{\kappa}\, A_3'(s) + A_3(s) &= A_3(s)\, \frac{(1-\beta)}{(1+\lambda_c\kappa)} \\ &\times [\cosh(\kappa s) + (\lambda_c\kappa)\sinh(\kappa s)]\, e^{-\kappa s} \\ &+ \frac{\beta U_0 \Psi_2}{\kappa\,(1+\lambda_c\kappa)} \Psi_6'(s)\, e^{-\kappa s}. \end{aligned} \qquad (31)$$

From Eq. (31) we obtain that at short distances, $s \ll \kappa^{-1}$, $A_3(s) \approx A_3(0)(1+\gamma_3 s)$, where

$$\gamma_3 = -\frac{\kappa}{1+\lambda_c\kappa} \left[\beta + \lambda_c\kappa - \frac{\beta U_0}{A_3(0)} \Psi_2 \frac{\Psi_6'(0)}{\kappa} \right]. \qquad (32)$$

From Eqs. (12) and (31) it follows that $\widetilde{u}(s+0;s) = A_3(s) + A_4(s) \approx A_0\,(1+\gamma s)$ at short distances, $s \ll \kappa^{-1}$, where

$$A_0 = A_3(0) + A_4(0) \qquad (33)$$

and

$$\gamma = [\gamma_3 A_3(0) - \kappa_2 A_4(0)]/A_0\,, \qquad (34)$$

while for long distances, $s \gg \kappa^{-1}$, $\widetilde{u}(s+0;s)$ decays exponentially,

$$\begin{aligned} &\widetilde{u}(s+0;s) \approx \mathcal{A}\, e^{-\kappa_2 s}, \\ &\mathcal{A} = \beta U_0 \left[\frac{\kappa}{2(\kappa+\kappa_2)} + \frac{\Psi_2}{(1+\beta-2\kappa_2/\kappa)} \right]. \end{aligned} \qquad (35)$$

The function $\widetilde{u}(s+0;s)$ may be approximated as

$$\widetilde{u}(s+0;s) \approx A_0\, \frac{(1+C)^\alpha\, e^{\kappa_3 s}}{(e^{\kappa_3 s}+C)^\alpha}\,, \qquad (36)$$

where

$$\alpha = 1 + \kappa_2/\kappa_3\,, \qquad (37)$$

$$C = (\kappa_2+\gamma)/(\kappa_3-\gamma)\,, \qquad (38)$$

and comparing Eqs. (34) and (36), we obtain a nonlinear equation, which defines the value κ_3:

$$\ln\frac{\mathcal{A}}{A_0} = \left(1+\frac{\kappa_2}{\kappa_3}\right) \ln\frac{\kappa_2+\kappa_3}{\kappa_3-\gamma}\,. \qquad (39)$$

Then, the IL stress ahead of the front is $\sigma_c(s) = k\,\widetilde{u}(s+0;s)/\lambda_c^2$, and the equation $\sigma_c(\Lambda) = \sigma_s$ defines the characteristic length Λ:

$$\Lambda \approx \kappa_3^{-1} \ln y\,, \tag{40}$$

where y is determined by the solution of the equation $\mathcal{B}y = (y+C)^\alpha$ with $\mathcal{B} = (1+C)^\alpha k A_0/(\sigma_s\lambda_c^2)$.

Using Eq. (35), Λ may approximately be presented as

$$\begin{aligned}\Lambda &\approx \kappa_2^{-1} \ln\left(k\mathcal{A}/\sigma_s\lambda_c^2\right) \\ &= \frac{1}{\kappa_2} \ln\left[\frac{2}{(1+\beta-2\kappa_2/\kappa)\,\Psi_1}\right].\end{aligned} \tag{41}$$

Equation (35) corresponds to the analytical solution for Λ, whereas Eq. (41) corresponds to the approximated analytical solution provided as Eq. 25 in the main text.